Survey of results on the ModPath and ModCycle problems

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This note summarizes the state of what is known about the tractability of the problem ModPath, which asks if an input undirected graph contains a simple st-path whose length satisfies modulo constraints. We also consider the problem ModCycle, which asks for the existence of a simple cycle subject to such constraints. We also discuss the status of these problems on directed graphs, and on restricted classes of graphs. We explain connections to the problem variant asking for a constant vertex-disjoint number of such paths or cycles, and discuss links to other related work.

1 Definition

A *simple path* in an undirected graph G is a path that does not visit the same vertex twice. (Note that we do not consider *trails*, which can reuse the same vertex twice but must not use the same edge twice.)

We study the following problem. It was posed in [MP22], though related questions have been studied much earlier (e.g., [Tho83, APY91]):

Definition 1.1. *Fix integers p and $q > 0$. Given an undirected graph G and two vertices s and t, we want to know if there exists a simple path connecting s and t in G whose length is $p \bmod q$. We call this the $\mathsf{ModPath}_{p,q}$ problem.*

A related problem is:

Definition 1.2. *Fix integers p and $q > 0$. Given an undirected graph G, we want to know if there is a simple cycle in G whose length is $p \bmod q$. We call this the $\mathsf{ModCycle}_{p,q}$ problem.*

These problems are clearly in NP, as we can easily check in polynomial time that a path or cycle is suitable. The question is whether they are in PTIME.

2 Easy reductions

Between cycles and paths. It is easy to see that the ModCycle problem reduces to the ModPath problem:

Proposition 2.1. *For any integers p and $q > 0$, the problem* $\mathsf{ModCycle}_{p,q}$ *reduces in PTIME (with a Turing reduction) to the problem* $\mathsf{ModPath}_{p-1,q}$.

Proof. We show how to reduce the problem, for each edge e of the graph, of determining whether there is a cycle satisfying the length constraint and using the edge e. This clearly suffices, as we can then simply test each possible choice of an edge e. For the reduction, we modify the graph G to a graph G_e where the edge e is removed, and where the source and sink to the ModPath problem are the endpoints of the edge e. Clearly there is a bijection between cycles in G using the edge e and st-paths in G_e, and this bijection maps cycles of length $p \bmod q$ to paths of length $p - 1 \bmod q$. □

We can show a similar reduction in the other direction, but the modulo is changed:

Proposition 2.2. *For any integers p and $q > 0$, the problem* $\mathsf{ModPath}_{p,q}$ *reduces in PTIME (with a Karp reduction) to the problem* $\mathsf{ModCycle}_{2p+1,2q}$.

Proof. Given the input undirected graph G to $\mathsf{ModPath}_{p,q}$ with source and sink s and t, subdivide each edge twice, and add a single edge connecting s and t. We let G' be the result. Given a path of length $p \bmod q$ connecting s and t in G, we deduce a path of length $2p \bmod 2q$ connecting s and t in G', hence a cycle of length $2p + 1 \bmod 2q$ in G' thanks to the extra edge. Conversely, we see that all cycles in G' have even length in G except if they use the additional edge between s and t, in which case they give us a simple path between s and t. If the cycle has length $2p + 1 \bmod 2q$ with the extra edge, then the simple path in question has length $2p \bmod 2q$ in G', hence length $p \bmod q$ in G. □

We are not aware of a reduction from ModPath to ModCycle which preserves the modulo.

On remainders. It is also clear that, for the ModPath problem, the value of the reminder does not matter:

Proposition 2.3. *For any integers p and p' and $q > 0$, the problem* $\mathsf{ModPath}_{p,q}$ *reduces in PTIME (with a Karp reduction) to the problem* $\mathsf{ModPath}_{p',q}$.

Proof. Simply add a path of the suitable length connecting t to a new vertex t', and reduce to $\mathsf{ModPath}_{p',q}$ with source s and target t'. □

Our results in Section 5 will imply that the same is not true of ModCycle (assuming that P is different from NP).

On moduli. It is also obvious that, for paths, the problem is at least as hard when taking a multiple of the original modulo.

Proposition 2.4. *For any integers p and $q > 0$ and $k > 0$, the problem* $\mathsf{ModPath}_{p,q}$ *reduces in PTIME (with a Turing reduction) to the problem* $\mathsf{ModPath}_{p,kq}$.

Proof. There is a path connecting s and t with length $p \bmod q$ iff there is a path of length $p+k'q \bmod kq$ for some k, so we can conclude using the oracle and using Proposition 2.3. ☐

A similar reduction works from $\mathsf{ModCycle}_{p,q}$ if we assume an oracle for the problems $\mathsf{ModCycle}_{p',kq}$ with $p' = p + iq$ for every i.

3 On directed graphs

In this section, we discuss the status of the problems ModPath and ModCycle when studying them on directed graphs instead of undirected graphs. We exclude the trivial case of the modulo $q = 1$ as the problems then amount to reachability or to testing the existence of a directed cycle which are clearly solvable in polynomial time.

Paths. The analogue of the ModPath problem on directed graphs is NP-hard. Indeed, the following is known (with an elementary but non-trivial proof):

Proposition 3.1 ([FHW80]). *The problem, given a directed graph G and vertices s, t, s', t', of deciding if there is a path from s to t and a path from s' to t' that are vertex-disjoint, is NP-hard.*

Proposition 3.1 implies that ModPath is hard on directed graphs:

Proposition 3.2. *Fix any p and $q \geq 2$. The problem, given a directed graph G and vertices s and t, of testing if there is a simple path of length p mod q from s to t, is NP-hard.*

Proof. We reduce from Proposition 3.1. We first assume $p > 0$. Then, given a directed graph G with vertices s, t, s', t', we replace each edge by a path of q edges, choose s as the source and t' as the sink, and add a path of p edges from t to s'. Then it is clear that any path of length $p \bmod q$ from s to t must use the path of p edges, and thus give a solution to the problem of Proposition 3.1. Conversely, any solution to the latter problem gives a path for the former problem.

If $p = 0$, we subdivide G as indicated, we choose as source a fresh vertex with a 1-edge path to s, choose t' as sink, and add a path of $q - 1$ edges from t to s'. The reasoning is similar. ☐

Cycles. Intractability also holds for the problem $\mathsf{ModCycle}$ on directed graphs in the case where $p \neq 0$ and q is large enough, as was already observed in [APY91]:

Proposition 3.3. *Fix any $0 < p < q$ and $q \geq 3$. The problem, given a directed graph G, of testing if there is a simple cycle of length $p \bmod q$, is NP-hard.*

Proof. We first show that there are two values $0 < p_1, p_2 < q$ such that $p \neq p_1$, $p \neq p_2$, and $p = p_1 + p_2 \bmod q$. If $p \geq 2$ then this is clear, taking $p_1 = 1$ and $p_2 = p_1 - 1$. If $p = 1$, take $p_1 = 2$ and $p_2 = q - 1$, noting that $p_1 < q$ and $p_2 \neq q$ because $q \geq 3$.

We reduce from Proposition 3.1 like in the first case of Proposition 3.2: given the directed graph G, we subdivide each edge to a path of length q, then add a directed path of length p_1 from t to s' and a directed path of length p_2 from t' to s. Now, from a solution to the problem of Proposition 3.1, we deduce a cycle of length $p_1+p_2 = p \bmod q$. Conversely, in G, we can partition the cycles among those that use either none of the extra paths, one of the extra paths, or both extra paths. Their modulo values are 0, p_1 or p_2, and p respectively. Now, as $p_1 \neq p$ and $p_2 \neq p$ and $p \neq 0$, this means that a cycle of length $p \bmod q$ must use both extra paths. This gives us two disjoint paths from s to t and from s' to t' in G, hence in the initial graph. $\square$

A more general complexity classification is given in [HST04] for a variant of the problem where the remainder modulo the value q is required to fall in a certain set S of allowed remainders (instead of $S = \{p\}$), provided that the set S of allowed remainders does not include 0.

For the case where the requested remainder is $p = 0$, i.e., the problem $\mathsf{ModCycle}_{0,q}$ with $q \geq 3$ on directed graphs, the complexity appears to be open: this is stated as open in [HST04]. In other words, for any fixed $q \geq 3$, it is open whether we can determine in PTIME, given a directed graph, whether it contains a simple cycle of length multiple of q. Also note that this same problem with $p = 0$ is known to tractable on undirected graphs (Section 5).

For the case $q = 2$, it is known that we can (easily) test in PTIME whether a directed graph contains an odd cycle [Tho85], and (less easily) whether it contains an even cycle [RST99, McC04]. Accordingly:

Proposition 3.4 ([Tho85, RST99, McC04])**.** *For $p \in \{0, 1\}$, the problem, given a directed graph G, of testing if there is a simple cycle of length $p \bmod 2$, is in PTIME.*

Note that this contrasts with the intractability of the same task for $\mathsf{ModPath}$ with $q = 2$ (Proposition 3.2).

Incidentally, only very recently was a tractable randomized algorithm shown to compute the *shortest* simple even cycle in a directed graph [BHK22]. The problem of the shortest simple odd cycle in a directed graph can easily be seen to be in polynomial time, thanks to the fact that the shortest odd cycle is necessarily simple [ch].

Restricted classes of directed graphs. One immediate observation is that all problems on directed graphs discussed in this section are tractable if the input is assumed to be a *directed acyclic graphs*: such graphs have no cycles, and the directed paths on such graphs are automatically simple.

4 Testing if all moduli are the same, and the case of modulo 2

Paths. It is shown in [APY91, Theorem 4] that, for any fixed p and q, one can test in PTIME, given a graph and vertices s and t, whether *all simple paths* connecting s and t in G have length $p \bmod q$. This implies in particular the following:

Proposition 4.1. *The problems* $\mathsf{ModPath}_{0,2}$ *and* $\mathsf{ModPath}_{1,2}$ *are in PTIME.*

The same result can be shown with a simpler proof due to Edmonds, see [LP84, Section II] for the case $p = 0$ and $q = 2$, which implies the case of $p = 1$ via Proposition 2.3.

Cycles. The same tractability result holds for the ModCycle problem. In fact, tractability even holds on directed graphs as we have seen (Proposition 3.4); but it can be shown to hold with an easy proof in the case of undirected graphs:

Proposition 4.2 (mentioned in [Tho85]). *The problems* $\mathsf{ModCycle}_{0,2}$ *and* $\mathsf{ModCycle}_{1,2}$ *are in PTIME.*

Proof. For undirected graphs and modulo two, an undirected graph has an odd cycle unless it is bipartite (which can be checked in PTIME), and it has an even cycle unless every biconnected component is an odd cycle or a single edge (which can be checked in PTIME). □

5 Cycles when the remainder is zero

For the problem ModCycle, when the requested remainder is 0, then the problem is known to be tractable:

Proposition 5.1 ([Tho88]). *For any* $q > 0$*, the problem* $\mathsf{ModCycle}_{0,q}$ *is in PTIME.*

This is because any large-treewidth graph must contain such a cycle. Specifically, [Tho88, Proposition 3.2] shows that any high-treewidth graph contains as topological minor a wall graph where all edges are $0 \bmod q$. Thus, the answer is yes on high-treewidth graphs, and on low-treewidth graphs the problems are always tractable (see Section 6).

6 Bounded-treewidth graphs

Under the assumption that the graphs have bounded treewidth, then the problem ModPath (hence, ModCycle by Proposition 2.1) is always in PTIME:

Proposition 6.1 (Theorem 5.2, [Tho88]). *Let* p *and* q *and* k *be arbitrary integers. The problem* $\mathsf{ModPath}_{p,q}$ *is in PTIME if we assume that the input graphs have treewidth at most* k*.*

Proof. One intuition is that we can process the graph along a tree decomposition and solve the problem by dynamic programming. Intuitively, we can remember, for each bag, for each set of disjoint pairs of endpoints and modulo lengths, which such sets are achievable simultaneously by disjoint paths in the subgraph induced by the nodes occurring below that bag in the tree decomposition. □

7 Constraints on degree and connectivity

In this section, we review some combinatorial results which study which conditions on the graph can guarantee the existence of cycles whose lengths achieve some prescribed remainder values. The conditions studied are minimal degree, average degree (or, equivalently, edge density), and connectivity.

Minimal degree. There are results showing that, when graphs are asserted to have sufficiently high minimal degree, then it is impossible to avoid some cycle lengths. Specifically, for graphs of sufficiently large ($O(q)$) minimal degree, then there must by cycles of all even lengths modulo q, and if the graph is 2-connected and not bipartite then there must be cycles of all lengths modulo q [GHLM22]. We note that the main result of [GHLM22] is actually a result about path lengths on graphs with sufficiently high minimal degree, so this also gives results on the ModPath problem in this setting.

Average degree. There are also existence results based on edge density [Bol77] (or equivalently on average degree). Specifically, for any odd modulo q, considering graphs with a sufficiently high number of edges (i.e., at least $c_q n$ some constant q, where n is the number of vertices), then such graphs must contain a cycle of length p mod q for every p. (Of course the same cannot be true for even q, e.g., considering bipartite graphs.) A similar result holds for all even remainders, i.e., cycles of length $2p$ modulo q for arbitrary q [Ver00].

Connectivity. There are also existence results based on connectivity. It is known that, for all $q \geq 3$, every q-connected graph contains a cycle of length zero modulo q [GHLM22]; note that the same is true assuming that the treewidth is sufficiently high (Section 5). Every q-connected graph must also contain cycles of all even lengths modulo q provided that $q \geq 6$ [GHLM22, Theorem 5.16]. Other results are known about the existence of k-linkage with modulo conditions assuming sufficiently high connectivity [CMZ09]. Last, it is shown in [LM21] that for *odd* moduli q, any sufficiently large 3-connected cubic graph contains cycles with each possible length modulo q.

Directed graphs. Some combinatorial results are also known for directed graphs. It is known that strongly connected directed graphs of sufficiently high edge density must contain an even cycle [CGK94]. Lower bounds on degree also imply the existence of cycles with remainder 0 in the case of directed graphs [AL89], and there are similar bounds on the dichromatic number [Ste22].

8 Multiple paths or multiple cycles

One natural generalization of ModPath and ModCycle is to ask for the existence of k disjoint paths or k disjoint cycles satisfying the conditions. These problems have been studied when k is given as input: in this case the problem is NP-hard for paths (as a special case of discrete multicommodity flow [Kar75]) and for cycles (already if we want to partition a graph on $3n$ vertices into n vertex-disjoint triangles; see [KH78] or [vRvKNB13]). They have also been studied when parameterizing by k, e.g., [BJK13]. Here we assume that k is a constant.

Without modulo constraints. The results of Robertson on Seymour imply that, on *undirected graphs*, for any constant k, given source-sink pairs $(s_1, t_1), \ldots, (s_k, t_k)$, we can decide in PTIME whether there exist k pairwise disjoint paths each of which connects s_i and t_i [RS95]. (Note that there are recent extensions of such results to problems where we want to minimize the *total length* of such paths, in the case $k = 2$ [BH19].) It is also known that we can test in linear time whether an input undirected graph contains k vertex-disjoint cycles [Bod94], note that this easily follows from a treewidth-based argument.

Now, on directed graphs, the existence of k vertex-disjoint cycles can also be tested in PTIME even on directed graphs [RRST96, Section 5]. By contrast, asking for the existence of two disjoint paths for two source-target pairs is NP-hard (see Proposition 3.1). Note that this last result no longer holds if the graph is required to be *planar*: in this case, the disjoint path problems for any constant k can be solved in polynomial time [Sch94].

Also note that these results have been extended in the setting where we are looking for k-tuples of paths that must be *shortest paths* from s_i to t_i. This task is known to be tractable on undirected graphs [Loc21], and on directed planar graphs or on directed graphs with $k = 2$ [BK17].

With modulo constraints. The results on k disjoint paths and cycles on undirected graphs have been extended to test the existence of some constant number of disjoint cycles or paths of prescribed modulo values. It is known that on undirected graphs you can test in PTIME for the presence of a constant number of disjoint cycles of length divisible by some q [Tho88, Theorem 5.1]; note that this follows from the proof of Proposition 5.1, as the answer is always yes on graphs of sufficiently high treewidth. It was shown in [KR10] that you can test in PTIME for the existence of k vertex-disjoint odd cycles in undirected graphs. You can also test in PTIME for the existence of k vertex-disjoint paths connecting k source-sink pairs with prescribed parities [KRW11].

Very recently [KKKX23], a tractability result for cycles with parity constraints was shown on directed graphs: you can test in PTIME on an input directed graph whether it contains k vertex-disjoint odd cycles.

9 Other related work

Group-labeled graphs. Another model is that of group-labeled graphs, for which there is a *directed* and *undirected* setting. The setting of *directed group-labeled graphs* [KKY20], considers *directed* graphs where edges are labeled by an element of an abelian group: the value of a directed path is the composition of the labels of the edges traversed by the path. However, edges can then also be traversed in a reverse direction, in which case we compose by the inverse of their label.

Hence, except in cases like $q = 2$, the model of directed group-labeled graphs does not seem well-adapted to code the ModPath or ModCycle problems. Note that the undirected and directed settings are equivalent where all nonzero elements of the group have order two [TY23].

The setting of *undirected group-labeled graphs* [Wol10, Wol11] is closer to our problem. In this setting, the model considers *undirected* graphs with each edge again labeled by an element of an abelian group, and with the value of a path being the combination of the edge labels.

In the setting of undirected group-labeled graphs, it was recently shown [TY23] that, for any prime power q, the cycles of length $p \bmod q$ satisfy the Erdős-Pósa property for all p. There are other similar results in this model on the Erdős-Pósa property for paths with prescribed endpoints and modulo values. There are other earlier results on cycles [GHK$^+$21, GHK$^+$22]. However, this does not seem to imply any result on the complexity of detecting whether such cycles or paths are present in an input graph.

Robertson-Seymour with parity conditions. There is work aiming at generalizing Robertson-Seymour results with parity conditions [KRW11, Kaw13]. However, these do not seem to have been extended to moduli greater than 2, and our problems are known to be tractable for $q = 2$ (Section 4).

Graphs with large clique minors. It is known that graphs with large clique minors must contain certain subgraphs with edges interpreted as multiples of some value [AK21, DDS21].

Expanding graphs. It is known that expanders, aka expanding graphs, must contain cycles of all moduli [MS23].

Planar graphs. It is known that on cubic, 3-connected, planar graphs, between any two vertices of an undirected graph there must be paths of all remainders modulo $q = 3$, and such paths can be found in polynomial time [DP91].

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